

CMB Temperature Anisotropy from Broken Spatial Isotropy due to an Homogeneous Cosmological Magnetic Field

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We derive the cosmic microwave background temperature anisotropy two-point correlation function (including off-diagonal correlations) from broken spatial isotropy due to an arbitrarily oriented homogeneous cosmological magnetic field.

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I. INTRODUCTION

Improving cosmic microwave background (CMB) anisotropy measurements are starting to make it possible to reconstruct physical conditions in the early Universe, and thus to constrain modifications of the standard cosmological and particle physics models [1]. In particular, analyses of the WMAP data suggest tentative indications of broken large-scale (statistical) spatial isotropy, see Refs. [2] for early indications and Refs. [3] for more recent studies. Statistical large-scale spatial isotropy is a major assumption of the standard cosmological model and has been well tested on length scales smaller than are probed by the large-scale CMB anisotropy data (see Sec. 3 of Ref. [4]). It is therefore important to understand if the larger-scale CMB anisotropy data really indicates that large-scale statistical spatial isotropy is broken [5]. This is part of the general program of testing for CMB anisotropy non-gaussianity.¹ In the last few years there has been much discussion of the “low” measured CMB temperature anisotropy quadrupole moment, the asymmetry between the CMB temperature anisotropy measured in the north and the south, the possibility of residual systematics and foreground emission in the data, etc. In addition to Refs. [1, 2, 3, 5], for early discussions of some of these issues see Refs. [8], for more recent discussions see Refs. [9]. The “low” measured quadrupole moment was also seen in the COBE-DMR data [10], while on smaller scales the CMB anisotropy is consistent with gaussianity [11].

There have been several theoretical attempts to explain the CMB temperature anisotropy large-scale anomalies as manifestations of departure from the standard cosmological scenario, e.g., via modifications of the inflation framework, in slightly anisotropic cosmological models, or by a preferred direction in the Universe, etc. See Refs. [12] for recent studies and Refs. [13] for earlier works. Recently Refs. [14] propose a cosmological magnetic field as a possible mechanism to explain these anomalies (the CMB temperature anisotropy non-gaussianity that results from the magnetic field presence has been used to limit the amplitude of such a field [15]).

In this paper we present a formalism useful for describing CMB temperature anisotropies in a cosmological model with a preferred direction at the perturbation level, while the background model preserves spatial isotropy. More specifically, we consider a cosmological model with a uniform magnetic field pointing in a fixed direction², with the magnetic field energy density treated as a first order perturbation, and study the CMB temperature anisotropy two-point correlation function which reflects the magnetic-field-induced broken spatial isotropy. A simplified version of this problem has been studied in Ref. [17]; here we consider an arbitrarily oriented magnetic field. In general, a cosmological magnetic field contributes, via the linearized Einstein equations, to all three kinds of perturbations,

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¹ See the Refs. [6] for reviews of non-gaussian models. In the simplest inflation models, quantum-mechanical zero-point fluctuations in a weakly coupled scalar field during inflation provide the initial conditions [7] for a gaussian CMB anisotropy, but non-gaussian initial conditions are possible in other inflation models.

² Such a magnetic field can be viewed as an approximation of a stochastic magnetic field with correlation length larger than the Hubble radius. A 10^{-9} Gauss cosmological magnetic field with correlation length larger than the Hubble radius can be generated by quantum-mechanical fluctuations during inflation, [16].

scalar, vector, and tensor, and if the amplitude of the magnetic field is large enough (10^{-9} G, or larger), there are observable imprints on the CMB temperature anisotropies (for recent reviews see Refs. [18]; for specific recent computations see Refs. [19]). As noted below, in our computation we only need to consider vector perturbations.

In the model we consider here the CMB temperature two-point correlation function reflects the presence of non-zero off-diagonal correlations between the usual a_{lm} multipole coefficients with multipole index l differing by 2 and/or multipole index m differing by 1 or 2. More precisely, there are non-zero off-diagonal correlations only for $\Delta l = \pm 2$ and $\Delta m = 0$ and for $\Delta m = \pm 1$ and ± 2 for both $\Delta l = 0$ and $\Delta l = \pm 2$. Some of these correlations have been discussed in Ref. [17] for the case of an homogeneous magnetic field oriented perpendicular to the galactic plane. Here we study the general case of an arbitrarily oriented magnetic field, and develop a new technique to compute the CMB temperature anisotropy in real space. The arbitrarily oriented magnetic field induces additional effects, not only breaking rotational invariance breaking (resulting in non-zero correlations between multipoles of different l), but also breaks spin (parity) symmetry (resulting in non-zero off-diagonal correlations between multipoles of different m). As a result, in multipole space the $\langle a_{lm}^* a_{l'm'} \rangle$ power spectrum is antisymmetric under exchange of m and m' . A similar effect occurs for Faraday rotation of the CMB polarization plane induced by an homogeneous magnetic field [20], for the cross-correlations between E -polarization anisotropy and temperature or B -polarization anisotropy, which vanish in the standard cosmological model in the absence of a primordial magnetic field.

The outline of our paper is as follows. In Sec. II we present the general description of the problem, that includes a derivation of the equations governing vorticity perturbations in the Universe (Sec. II.A) and an expression for the CMB temperature anisotropy induced by Alfvén waves (Sec. II.B). In Sec. III we derive the multipole coefficient power spectrum, which includes various $\Delta l = 0$, $\Delta l = \pm 2$, $\Delta m = 0$, $\Delta m = \pm 1$, and $\Delta m = \pm 2$ correlations. In Sec. IV we derive the real-space two-point temperature anisotropy correlation function (the details of the computation are summarized in App. B). We conclude in Sec. V. In App. A we list useful mathematical formulae that we used in the computations.

II. GENERAL DESCRIPTION

A. Vorticity perturbations

In this subsection we study the dynamics of linear magnetic vector perturbations about a spatially-flat³ Friedmann-Lemaître-Robertson-Walker (FLRW) homogeneous cosmological spacetime background with vector metric fluctuations. The metric tensor can be decomposed into a spatially homogeneous background part and a perturbation part, $g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}$, where $\mu, \nu \in (0, 1, 2, 3)$ are spacetime indices. For a spatially-flat model, and working with conformal time η , the background FLRW metric tensor $\bar{g}_{\mu\nu} = a^2 \eta_{\mu\nu}$, where $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ is the Minkowski metric tensor and $a(\eta)$ the scale factor. Vector perturbations are gauge dependent because the mapping of coordinates between the perturbed physical manifold and the background is not unique. Vector perturbations to the geometry can be described by two three-dimensional divergence-free vector fields \mathbf{A} and \mathbf{H} [22], where

$$\delta g_{0i} = \delta g_{i0} = a^2 A_i, \quad \delta g_{ij} = a^2 (H_{i,j} + H_{j,i}). \quad (1)$$

Here a comma denotes the usual spatial derivative, $i, j \in (1, 2, 3)$ are spatial indices, and \mathbf{A} and \mathbf{H} vanish at spatial infinity. Studying the behavior of these variables under infinitesimal general coordinate transformations (gauge transformations in the context of linearized gravity) one find that $\mathbf{V} = \mathbf{A} - \dot{\mathbf{H}}$ is gauge-invariant (the overdot represents a derivative with respect to conformal time). \mathbf{V} is a vector perturbation of the extrinsic curvature [23]. Exploiting the gauge freedom we choose \mathbf{H} to be constant in time. Then the vector metric perturbation may be described in terms of two divergenceless three-dimensional gauge-invariant vector fields, the vector potential \mathbf{V} and a vector representing the transverse peculiar velocity of the plasma, the vorticity $\mathbf{\Omega} = \mathbf{v} - \mathbf{V}$, where \mathbf{v} is the spatial part of the four-velocity perturbation of a stationary fluid element [24].⁴ In the absence of a source the vector perturbation \mathbf{V} decays with time (this follows from $\dot{\mathbf{V}} + 2(\dot{a}/a)\mathbf{V} = 0$) and so can be ignored.

Since the fluid velocity is small the displacement current in Ampère's law may be neglected; this implies the current \mathbf{J} is determined by the magnetic field via $\mathbf{J} = \nabla \times \mathbf{B}/(4\pi)$. The residual ionization of the primordial plasma is large enough to ensure that magnetic field lines are frozen into the plasma so the induction law takes the form

³ Current observational data are consistent with flat spatial hypersurfaces, see Ref. [21] for a recent review.

⁴ Given the general coordinate transformation properties of the velocity field \mathbf{v} , two gauge-invariant quantities can be constructed, the shear $\mathbf{s} = \mathbf{v} - \dot{\mathbf{H}}$ and the vorticity $\mathbf{\Omega} = \mathbf{v} - \mathbf{A}$ [23]. In the gauge $\dot{\mathbf{H}} = 0$ (i.e., $\mathbf{V} = \mathbf{A}$) we get $\mathbf{\Omega} = \mathbf{v} - \mathbf{V}$ [25].

$\dot{\mathbf{B}} = \nabla \times (\mathbf{v} \times \mathbf{B})$. As a result the baryon Euler equation for \mathbf{v} has the Lorentz force $\mathbf{L}(\mathbf{x}) = -\mathbf{B}(\mathbf{x}) \times [\nabla \times \mathbf{B}(\mathbf{x})] / (4\pi)$ as a source term. The photons are neutral so the photon Euler equation does not have a Lorentz force source term. The Euler equations for photons and baryons are [24, 26, 27]

$$\dot{\Omega}_\gamma + \dot{\tau}(\mathbf{v}_\gamma - \mathbf{v}_b) = 0, \quad (2)$$

$$\dot{\Omega}_b + \frac{\dot{a}}{a}\Omega_b - \frac{\dot{\tau}}{R}(\mathbf{v}_\gamma - \mathbf{v}_b) = \frac{\mathbf{L}^{(V)}(\mathbf{x})}{a^4(\rho_b + p_b)}, \quad (3)$$

where the subscripts γ and b refer to the photon and baryon fluids, and ρ and p are energy density and pressure. Here $\dot{\tau} = n_e \sigma_T a$ is the differential optical depth, n_e is the free electron density, σ_T is the Thomson cross section, $R = (\rho_b + p_b)/(\rho_\gamma + p_\gamma) \simeq 3\rho_b/4\rho_\gamma$ is the momentum density ratio between baryons and photons, and $L_i^{(V)}$ is the transverse vector (divergenceless) part of the Lorentz force. In the tight-coupling limit $\mathbf{v}_\gamma \simeq \mathbf{v}_b$, so we introduce the photon-baryon fluid divergenceless vorticity $\Omega (= \Omega_\gamma = \Omega_b)$ that satisfies

$$(1 + R)\dot{\Omega} + R\frac{\dot{a}}{a}\Omega = \frac{\mathbf{L}^{(V)}(\mathbf{x})}{a^4(\rho_\gamma + p_\gamma)}. \quad (4)$$

As usual we consider an expansion about a spatially homogeneous background magnetic field strength \mathbf{B}_0 , writing the total magnetic field $\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1$, where \mathbf{B}_1 is a small ($|\mathbf{B}_1| \ll |\mathbf{B}_0|$) first order inhomogeneous magnetic field strength perturbation that is divergenceless ($\nabla \cdot \mathbf{B}_1 = 0$). To leading order in \mathbf{B}_1 the induction law then gives

$$\dot{\mathbf{B}}_1 = \nabla \times \mathbf{v} \times \mathbf{B}_0. \quad (5)$$

The current in this case is determined by the magnetic field perturbation, $\mathbf{J} = \nabla \times \mathbf{B}_1 / (4\pi)$. Consequently the Lorentz force is $\mathbf{L}(\mathbf{x}) = -\mathbf{B}_0 \times [\nabla \times \mathbf{B}_1] / (4\pi)$.

Neglecting viscosity, which is a good approximation on scales much larger than the Silk damping length scale, taking the time derivative of Eq. (4), for a fixed Fourier⁵ mode \mathbf{k} , we get for the transverse vorticity

$$\ddot{\Omega} = \frac{-\mathbf{B}_0 \cdot \mathbf{k}^2}{4\pi(\rho_{\gamma 0} + p_{\gamma 0})}\Omega \quad (7)$$

in the radiation dominated epoch when $R \ll 1$. Here $\rho_{\gamma 0}$ and $p_{\gamma 0}$ denote the present value of the photon energy density and pressure and we have used $\dot{R}/R = \dot{a}/a$. In general, the factor $1 + R$ appearing in Eq. (4) leads to the suppression of the vorticity amplitude due to the tight coupling between photons and baryons, because photons being neutral are not affected by the Lorentz force. This suppression happens only for scales larger than the Silk damping length scale, leaving the amplitude of vorticity perturbations unchanged for $k > k_S$ (k_S is the wavenumber corresponding to the Silk damping length scale) [28].

Equation (7) describes Alfvén wave propagation in the expanding Universe. These Alfvén waves propagate with phase velocity $v_A \mathbf{b} \cdot \hat{\mathbf{k}} = v_A \mu$, where the Alfvén velocity $v_A = B_0 / \sqrt{4\pi(\rho_{\gamma 0} + p_{\gamma 0})}$, $\mathbf{b} = \mathbf{B}_0 / B_0$ is the unit vector in the direction of the magnetic field, and $\hat{\mathbf{k}}$ is the unit wavevector in the propagation direction. Equation (7) has two independent solutions, conventionally picked to be cos and sin functions. The cos solution describes vector perturbations in the absence of the magnetic field and thus is not of interest here. The sin solution [$\propto \sin(v_A k \mu \eta + \phi)$, where ϕ is a constant of integration] describes transverse Alfvén waves. For a finite vorticity-energy-density, vorticity must vanish on super-Hubble-radius scales ($k\eta \rightarrow 0$), $\Omega(k\eta \rightarrow 0) \rightarrow 0$, which implies $\phi = 0$, so the solution of Eq. (7) is [17]

$$\Omega(\mathbf{k}, \eta) = \Omega_0 \sin(v_A k \eta \mu), \quad (8)$$

where Ω_0 is the initial amplitude of the vorticity perturbation in the fluid. Self-consistency⁶ requires $|\Omega_0| = |\mathbf{B}_1| v_A / |\mathbf{B}_0| = |\mathbf{B}_1| / \sqrt{4\pi(\rho_{\gamma 0} + p_{\gamma 0})}$, allowing an initial vorticity amplitude a factor $|\mathbf{B}_1| / |\mathbf{B}_0|$ ($\ll 1$) smaller than the

⁵ For a vector field \mathbf{F} we use

$$F_j(\mathbf{k}) = \int d^3x e^{i\mathbf{k} \cdot \mathbf{x}} F_j(\mathbf{x}), \quad F_j(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} e^{-i\mathbf{k} \cdot \mathbf{x}} F_j(\mathbf{k}), \quad (6)$$

when Fourier transforming between real and wavenumber spaces; we assume flat spatial hypersurfaces.

⁶ In terms of the magnetic field perturbation we have

$$\dot{\Omega} = \frac{i(\mathbf{B}_0 \cdot \mathbf{k})}{4\pi(\rho_{\gamma 0} + p_{\gamma 0})} \mathbf{B}_1 = i v_A^2 \mu k \frac{\mathbf{B}_1}{|\mathbf{B}_0|}. \quad (9)$$

It is easy to see that Ω_0 is directed along \mathbf{B}_1 , and using $\Omega = \Omega_0 \exp(i v_A k \mu \eta + i\phi)$, we obtain $i|\Omega_0| v_A k \mu = i v_A^2 \mu k |\mathbf{B}_1| / |\mathbf{B}_0|$.

Alfvén velocity. Thus, Alfvén wave excitations in the Universe require (i) initial vector (vorticity) perturbations, and (ii) a cosmological background magnetic field. Since v_A is treated as a $1/2$ -order perturbation, and the inhomogeneous magnetic field is a first-order perturbation ($|\mathbf{B}_1| \ll |\mathbf{B}_0|$), the amplitude of the vorticity perturbation is a $3/2$ -order perturbation.

We assume that the initial vorticity perturbation spectrum in wavenumber space is that of a stochastic gaussianly-distributed vector field with helicity [27],

$$\langle \Omega_{0,i}^*(\mathbf{k}) \Omega_{0,j}(\mathbf{k}') \rangle = (2\pi)^3 \delta^{(3)}(\mathbf{k} - \mathbf{k}') [P_{ij}(\hat{\mathbf{k}}) P_{\Omega_0}(k) + i \epsilon_{ijl} \hat{k}_l P_{H_0}(k)]. \quad (10)$$

Here $P_{ij}(\hat{\mathbf{k}}) = \delta_{ij} - \hat{k}_i \hat{k}_j$ is the transverse plane projector with unit wavenumber components $\hat{k}_i = k_i/k$, a star denotes complex conjugation, ϵ_{ijl} is the antisymmetric tensor, and $\delta^{(3)}(\mathbf{k} - \mathbf{k}')$ is the Dirac delta function. The power spectra $P_{\Omega_0}(k)$ and $P_{H_0}(k)$ determine the initial kinetic energy density and average helicity of vortical motions. We approximate both spectra by simple power laws with indices n_Ω and n_H .

B. CMB temperature anisotropies from Alfvén waves

Our aim is to study CMB temperature anisotropies $\Delta T/T(\mathbf{n}, \mathbf{x}_0, \eta_0)$ in the presence of an homogeneous cosmological magnetic field \mathbf{B} . As usual, $\Delta T = T(\mathbf{n}, \mathbf{x}_0, \eta_0) - \bar{T}$, where \bar{T} is the mean temperature, \mathbf{n} is the unit vector in the photon arrival direction, \mathbf{x}_0 is the position of the observer, and η_0 is current conformal time (since the big bang).

Vector perturbations induce CMB temperature anisotropies via the Doppler and integrated Sachs-Wolfe effects [17],

$$\frac{\Delta T}{T}(\eta_0, \mathbf{n}) = -\mathbf{v} \cdot \mathbf{n} \Big|_{\eta_{\text{dec}}}^{\eta_0} + \int_{\eta_{\text{dec}}}^{\eta_0} d\eta \dot{\mathbf{V}} \cdot \mathbf{n}, \quad (11)$$

where η_{dec} is the conformal time at decoupling. The decaying nature of the vector potential \mathbf{V} implies that most of its contribution toward the integrated Sachs-Wolfe term comes from near η_{dec} . Neglecting a possible dipole contribution due to \mathbf{v} today, we obtain [17],

$$\frac{\Delta T}{T}(\eta_0, \mathbf{n}) \simeq \mathbf{v}(\eta_{\text{dec}}) \cdot \mathbf{n} - \mathbf{V}(\eta_{\text{dec}}) \cdot \mathbf{n} = \boldsymbol{\Omega}_0 \cdot \mathbf{n} \quad (12)$$

(where $\boldsymbol{\Omega}_0 = \boldsymbol{\Omega}(\eta_{\text{dec}})$), leading to [17],

$$\frac{\Delta T}{T}(\mathbf{n}, \mathbf{k}, \eta_0) = v_A k \eta_{\text{dec}} \mu (\boldsymbol{\Omega}_0(\mathbf{k}) \cdot \mathbf{n}) e^{i\mathbf{k} \cdot \mathbf{n} \Delta \eta}, \quad (13)$$

where wavevector $\mathbf{k} = k\hat{\mathbf{k}}$ labels the resulting Fourier mode after transforming from the coordinate representation \mathbf{x}_0 to the momentum representation by using $e^{i\mathbf{k} \cdot \mathbf{x}_0}$, and $\Delta \eta = \eta_0 - \eta_{\text{dec}} \approx \eta_0$ is the conformal time from decoupling until today.

To compute $\langle \Delta T/T(\mathbf{n}) \Delta T/T(\mathbf{n}') \rangle$ we can follow Ref. [17], but the computation is simpler if we introduce vector spherical harmonics [29]. Using the decomposition into vector spherical harmonics,

$$\boldsymbol{\Omega}_0(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{n} \Delta \eta} = \sum_{l,\lambda,m} A_{lm}^{(\lambda)}(\mathbf{k}) \mathbf{Y}_{lm}^{(\lambda)}(\mathbf{n}), \quad (14)$$

where $\mathbf{Y}_{lm}^{(\lambda)}(\mathbf{n})$ (with $\lambda = -1, 0, 1$) are vector spherical harmonics (see Eqs. (A8) below for definitions), and $A_{lm}^{(\lambda)}$ are decomposition coefficients, and taking into account the relations $\sum_{\lambda} \mathbf{n} \cdot \mathbf{Y}_{lm}^{(\lambda)}(\mathbf{n}) = \mathbf{n} \cdot \mathbf{Y}_{lm}^{(-1)}(\mathbf{n}) = Y_{lm}(\mathbf{n})$ (see Eq. (72), p. 220, [29], where $Y_{lm}(\mathbf{n})$ are the usual spherical harmonics), we obtain

$$\frac{\Delta T}{T}(\mathbf{n}, \mathbf{k}, \eta_0) = v_A k \eta_{\text{dec}} \mu \sum_{l,m} A_{lm}^{(-1)} Y_{lm}(\mathbf{n}). \quad (15)$$

Comparing to the conventional spherical harmonic decomposition, $\Delta T/T(\mathbf{n}, \mathbf{k}, \eta_0) = \sum_{l,m} a_{lm}(\mathbf{k}, \eta_0) Y_{lm}(\mathbf{n})$, makes it possible to relate the usual a_{lm} multipole coefficients to $A_{lm}^{(-1)}$,

$$a_{lm}(\mathbf{k}) = v_A k \eta_{\text{dec}} \mu A_{lm}^{(-1)}(\mathbf{k}). \quad (16)$$

Information about the $\Omega_0(\mathbf{k})$ spectrum is encoded in the $A_{lm}^{(-1)}$ coefficients, which (using Eq. (135), p. 229, [29]) can be expressed as

$$A_{lm}^{(-1)}(\mathbf{k}) = 4\pi i^{l-1} \frac{\sqrt{l(l+1)}}{2l+1} [j_{l-1}(k\eta_0) + j_{l+1}(k\eta_0)] \Omega_0(\mathbf{k}) \cdot \mathbf{Y}_{lm}^{(+1)\star}(\hat{\mathbf{k}}). \quad (17)$$

Here $j_l(x)$ are spherical Bessel functions and we have omitted a term $\propto \Omega_0(\mathbf{k}) \cdot \mathbf{Y}_{lm}^{(-1)\star}(\hat{\mathbf{k}})$ because the vorticity vector field is transverse, $\mathbf{k} \cdot \Omega_0(\mathbf{k}) = 0$, and so $\Omega_0(\mathbf{k}) \cdot \mathbf{Y}_{lm}^{(-1)\star}(\hat{\mathbf{k}}) = 0$.

We are now in a position to compute the $\langle a_{lm}^* a_{l'm'} \rangle$ power spectrum,

$$\begin{aligned} \langle a_{lm}^* a_{l'm'} \rangle &= \frac{1}{(2\pi)^3} \int dk k^2 d\Omega_{\hat{\mathbf{k}}} a_{lm}^*(\mathbf{k}) a_{l'm'}(\mathbf{k}) \\ &= \frac{2i^{l'-l}}{\pi} \sqrt{l'l'(l+1)(l'+1)} \int dk k^2 P_{\Omega_0}(k) v_A^2 \left(\frac{\eta_{\text{dec}}}{\eta_0} \right)^2 j_l(k\eta_0) j_{l'}(k\eta_0) \sum_{i,j=1}^3 P_{ij} \int d\Omega_{\hat{\mathbf{k}}} |\mu|^2 |\mathbf{Y}_{lm}^{(+1)}(\hat{\mathbf{k}})|^i |\mathbf{Y}_{l'm'}^{(+1)\star}(\hat{\mathbf{k}})|^j, \end{aligned} \quad (18)$$

where $d\Omega_{\hat{\mathbf{k}}}$ represents the solid angle volume element, $\mu = \mathbf{b} \cdot \hat{\mathbf{k}}$, and we have used Eq. (10). It can be shown that initial helicity does not contribute to the integral in Eq. (18) (see Sec. III of Ref. [30]). Performing the sum over i and j (we use Eqs. (74), p. 220, [29], and vector spherical harmonics properties listed in App. A.2 below) results in

$$\begin{aligned} \langle a_{lm}^* a_{l'm'} \rangle &= \frac{2i^{l'-l}}{\pi} \sqrt{l'l'(l+1)(l'+1)} \int dk k^2 P_{\Omega_0}(k) v_A^2 \left(\frac{\eta_{\text{dec}}}{\eta_0} \right)^2 j_l(k\eta_0) j_{l'}(k\eta_0) \\ &\quad \times \int d\Omega_{\hat{\mathbf{k}}} \left\{ \mathbf{Y}_{lm}^{(+1)}(\hat{\mathbf{k}}) \cdot \mathbf{Y}_{l'm'}^{(+1)\star}(\hat{\mathbf{k}}) - (\mathbf{b} \cdot \mathbf{Y}_{lm}^{(+1)}(\hat{\mathbf{k}})) (\mathbf{b} \cdot \mathbf{Y}_{l'm'}^{(+1)\star}(\hat{\mathbf{k}})) - (\mathbf{b} \cdot \mathbf{Y}_{lm}^{(0)}(\hat{\mathbf{k}})) (\mathbf{b} \cdot \mathbf{Y}_{l'm'}^{(0)\star}(\hat{\mathbf{k}})) \right\}. \end{aligned} \quad (19)$$

An advantage of this computational method over that of Ref. [17] is that in Eq. (18) we didn't need to integrate over $d\Omega_{\mathbf{n}}$ and $d\Omega_{\mathbf{n}'}$. This is similar to what happens in the total angular momentum method [25].

As a consequence of the orthonormality relation,

$$\int d\Omega_{\hat{\mathbf{k}}} \mathbf{Y}_{lm}^{(\lambda)}(\hat{\mathbf{k}}) \cdot \mathbf{Y}_{l'm'}^{(\lambda')\star}(\hat{\mathbf{k}}) = \delta_{\lambda\lambda'} \delta_{ll'} \delta_{mm'}, \quad (20)$$

the first term in the $d\Omega_{\hat{\mathbf{k}}}$ integral in Eq. (19) results in the usual diagonal correlations. The second term in Eq. (19) includes non-zero correlations for $l = l'$ and $l = l' \pm 2$, as well as $m = m'$, $m = m' \pm 1$, and $m = m' \pm 2$ (if $\mathbf{b} \parallel \mathbf{z}$ there are non-zero correlations only for $m = m'$ [17]), while the third term includes non-zero correlations for $l = l'$ and $m = m'$, $m = m' \pm 1$, and $m = m' \pm 2$ (again, if $\mathbf{b} \parallel \mathbf{z}$ there are non-zero correlations only for $m = m'$ [17]).

To simplify the computation, we rewrite the last two terms in the $d\Omega_{\hat{\mathbf{k}}}$ integral in Eq. (19) in terms of Wigner D functions. Wigner D functions relate helicity basis vectors $\mathbf{e}'_{\pm 1} = \mp(\mathbf{e}_\Theta \pm i\mathbf{e}_\phi)/\sqrt{2}$ and $\mathbf{e}'_0 = \mathbf{e}_r$ to spherical basis vectors $\mathbf{e}_{\pm 1} = \mp(\mathbf{e}_x \pm i\mathbf{e}_y)/\sqrt{2}$ and $\mathbf{e}_0 = \mathbf{e}_z$ (see Eq. (53), p. 11, [29]) through

$$\mathbf{e}'_\mu = \sum_\nu D_{\nu\mu}^1(\phi, \Theta, 0) \mathbf{e}_\nu, \quad \nu, \mu = -1, 0, 1. \quad (21)$$

In both the spherical basis and the helicity basis the following relations hold: $\mathbf{e}_\nu \mathbf{e}^\mu = \delta_{\nu\mu}$, $\mathbf{e}^\mu = (-1)^\mu \mathbf{e}_{-\mu}$, $\mathbf{e}^\mu = \mathbf{e}_\mu^*$, $\mathbf{e}_\mu \times \mathbf{e}_\nu = -i\epsilon_{\mu\nu\lambda} \mathbf{e}_\lambda$.

Vector spherical harmonics may be expressed in terms of Wigner D functions in the helicity basis where the angles Θ and ϕ are defined in terms of the unit wavevector $\hat{\mathbf{k}}$, see Eqs. (A18). Using these relations the last two terms in the $d\Omega_{\hat{\mathbf{k}}}$ integral in Eq. (19) become

$$\begin{aligned} &\frac{-1}{4\pi} \sqrt{(2l+1)(2l'+1)} \left[(\mathbf{b} \cdot \mathbf{e}'_{+1}(\Theta, \phi)) (\mathbf{b} \cdot \mathbf{e}'_{+1}(\Theta, \phi))^* D_{-1,-m}^l(0, \Theta, \phi) D_{-1,-m'}^{l'\star}(0, \Theta, \phi) \right. \\ &\quad \left. + (\mathbf{b} \cdot \mathbf{e}'_{-1}(\Theta, \phi)) (\mathbf{b} \cdot \mathbf{e}'_{-1}(\Theta, \phi))^* D_{1,-m}^l(0, \Theta, \phi) D_{1,-m'}^{l'\star}(0, \Theta, \phi) \right]. \end{aligned} \quad (22)$$

The unit vector field \mathbf{b} may be written in terms of spherical harmonics (see Eq. (13), p. 13, [29]), and using Eqs. (21), (22), and (A19), we obtain for the $d\Omega_{\hat{\mathbf{k}}}$ integral in Eq. (19),

$$\begin{aligned} &\int d\Omega_{\hat{\mathbf{k}}} \left\{ \mathbf{Y}_{lm}^{(+1)}(\hat{\mathbf{k}}) \cdot \mathbf{Y}_{l'm'}^{(+1)\star}(\hat{\mathbf{k}}) - (\mathbf{b} \cdot \mathbf{Y}_{lm}^{(+1)}(\hat{\mathbf{k}})) (\mathbf{b} \cdot \mathbf{Y}_{l'm'}^{(+1)\star}(\hat{\mathbf{k}})) - (\mathbf{b} \cdot \mathbf{Y}_{lm}^{(0)}(\hat{\mathbf{k}})) (\mathbf{b} \cdot \mathbf{Y}_{l'm'}^{(0)\star}(\hat{\mathbf{k}})) \right\} \\ &= \delta_{ll'} \delta_{mm'} - \frac{2\pi}{3} \{1 + (-1)^{l+l'}\} \sqrt{(2l+1)(2l'+1)} \int_0^\pi d\Theta \sin\Theta \sum_{\nu,\nu'=-1}^1 (-1)^{\nu+\nu'} \delta_{m,m'-\nu+\nu'} Y_{1,\nu}^*(\mathbf{b}) Y_{1,\nu'}(\mathbf{b}) \\ &\quad \times d_{-\nu,1}^1(\Theta) d_{-\nu',1}^1(\Theta) d_{-1,-m}^l(\Theta) d_{-1,-m'}^{l'}(\Theta). \end{aligned} \quad (23)$$

Here the $d_{mm'}^l(\beta)$ functions are defined in Sec. 4.3 of Ref. [29], and we have used the reality of these functions as well as the relations $d_{m,m'}^l(\pi - \Theta) = (-1)^{l-m'} d_{-m,m'}^l(\Theta) = (-1)^{l+m} d_{m,-m'}^l(\Theta)$ (Eq. (1), p. 79, [29]). The expression in Eq. (23) indicates that there are in general nonzero correlations for $l = l' \pm a$, where a is even. In addition there are the following possibilities: (i) when $\nu = \nu'$ there are nonzero correlations for $m = m'$; (ii) when $|\nu - \nu'| = 1$ there are nonzero correlations for $m = m' \pm 1$; and (iii) when $|\nu - \nu'| = 2$ there are nonzero correlations for $m = m' \pm 2$.

It is convenient to introduce the notation

$$I_d^{(l,l')} = \frac{2}{\pi} \int dk k^2 P_{\Omega_0}(k) v_A^2 \left(\frac{\eta_{\text{dec}}}{\eta_0} \right)^2 j_l(k\eta_0) j_{l'}(k\eta_0). \quad (24)$$

Then Eq. (23), for the multipole coefficients power spectrum, may be rewritten as

$$\begin{aligned} \langle a_{lm}^* a_{l'm'} \rangle &= i^{l-l'} \sqrt{ll'(l+1)(l'+1)} I_d^{(l,l')} \\ &\times \left[\delta_{ll'} \delta_{mm'} - 2\pi \{1 + (-1)^{l+l'}\} \sqrt{(2l+1)(2l'+1)} \int_0^\pi d\Theta \sin \Theta S_{mm'}(\Theta, \Theta_B, \phi_B) d_{-1,-m}^l(\Theta) d_{-1,-m'}^{l'}(\Theta) \right], \end{aligned} \quad (25)$$

where we have defined

$$S_{mm'}(\Theta, \Theta_B, \phi_B) = \frac{1}{3} \sum_{\nu, \nu'=-1}^1 (-1)^{\nu+\nu'} Y_{1,\nu}^*(\mathbf{b}) Y_{1,\nu'}(\mathbf{b}) \delta_{m,m'-\nu+\nu'} d_{-\nu,1}^1(\Theta) d_{-\nu',1}^1(\Theta). \quad (26)$$

For $l + l'$ odd the $1 + (-1)^{l+l'}$ factor in Eq. (25) is zero and so is the off-diagonal piece. For $l + l'$ even we must sum over ν and ν' in Eq. (26). Using expressions for $d_{\nu,1}^l(\Theta)$ (Eq. (16), p. 78, [29]) and $Y_{1,\nu}(\mathbf{b})$ (Eq. (2), p. 155, [29]), the double summation results in 5 different terms (corresponding to nonzero correlations for $m = m'$, $m = m' \pm 1$, and $m = m' \pm 2$),

$$\begin{aligned} S_{mm'}(\Theta, \Theta_B, \phi_B) &= \frac{1}{16\pi} \left\{ [(1 + \cos^2 \Theta_B) - (3 \cos^2 \Theta_B - 1) \cos^2 \Theta] \delta_{mm'} \right. \\ &\quad - 2 \sin \Theta_B \cos \Theta_B \sin \Theta \cos \Theta [e^{-i\phi_B} \delta_{m,m'-1} + e^{i\phi_B} \delta_{m,m'+1}] \\ &\quad \left. - \frac{1}{2} \sin^2 \Theta_B \sin^2 \Theta [e^{-2i\phi_B} \delta_{m,m'-2} + e^{2i\phi_B} \delta_{m,m'+2}] \right\}. \end{aligned} \quad (27)$$

When $\mathbf{b} \parallel \mathbf{z}$, $\sin \Theta_B = \phi_B = 0$ and only one term survives,

$$S_{mm'}(\Theta, 0, 0) = \frac{1}{3} \sum_{\nu, \nu'=-1}^1 Y_{1,\nu}^*(\mathbf{b}) Y_{1,\nu'}(\mathbf{b}) \delta_{m,m'-\nu+\nu'} d_{-\nu,1}^1(\Theta) d_{-\nu',1}^1(\Theta) = \frac{1}{8\pi} (1 - \cos^2 \Theta) \delta_{mm'}. \quad (28)$$

This reproduces the result of Ref. [17].

III. MULTIPOLE COEFFICIENT POWER SPECTRUM

For an arbitrary \mathbf{B}_0 the multipole coefficient power spectrum is a function of two spherical angles, corresponding to the angular separation between \mathbf{b} and directional vectors \mathbf{n} and \mathbf{n}' . The amplitude of the power spectrum depends on v_A , P_{Ω_0} , and the photon travel distances from decoupling until today. In this Section we study diagonal (in terms of l) $l = l'$ and off-diagonal $l = l' \pm 2$ correlations separately. We note that the terms with $l = l'$ and $m = m'$ we compute here are purely due to the presence of the magnetic field and must be added to the usual CMB temperature anisotropy terms induced by other sources (for example, scalar and/or tensor perturbations generated by quantum fluctuations during inflation). Since the magnetic field amplitude is small we ignore correlations between magnetic field and scalar (or other) perturbations.

A. $l = l'$ correlations

For $l = l'$, the integral expression of Eq. (24) takes the form,

$$I_d^{(l,l)} = \frac{2}{\pi} \int dk k^2 P_{\Omega_0}(k) v_A^2 \left(\frac{\eta_{\text{dec}}}{\eta_0} \right)^2 j_l^2(k\eta_0). \quad (29)$$

The corresponding multipole coefficients power spectrum, Eq. (25), becomes,

$$\begin{aligned} C_l^{(m,m')}(\Theta_B, \phi_B) &= \langle a_{lm}^* a_{lm'} \rangle \\ &= \left\{ (3 \cos^2 \Theta_B - 1) I_{0,0} \delta_{mm'} + 2 \sin \Theta_B \cos \Theta_B \left[e^{-i\phi_B} \delta_{m,m'-1} I_{0,-1} + e^{i\phi_B} \delta_{m,m'+1} I_{0,+1} \right] \right. \\ &\quad \left. + \frac{1}{2} \sin^2 \Theta_B \left[e^{-2i\phi_B} \delta_{m,m'-2} I_{0,-2} + e^{2i\phi_B} \delta_{m,m'+2} I_{0,+2} \right] \right\} I_d^{(l,l)}, \end{aligned} \quad (30)$$

and for $\mathbf{b} \parallel \mathbf{z}$, $\sin \Theta_B = 0$ and it is easy to recover the result of Ref. [17], $C_l(m) = (3 \cos^2 \Theta_B - 1) I_{0,0} I_d^{(l,l)}$. In Eq. (30) the coefficient $I_{0,0}$ is

$$I_{0,0}(l, m, \Theta_B) = \frac{l(l+1)}{(2l-1)(2l+3)} \left\{ \frac{l(l+1) + (l^2 + l - 3) \cos^2 \Theta_B}{3 \cos^2 \Theta_B - 1} - m^2 \left[1 - \frac{3}{l(l+1)} \right] \right\}, \quad (31)$$

and

$$I_{0,\pm 1}(l, m) = -\frac{l^2 + l - 3}{(2l-1)(2l+3)} \left(m \mp \frac{1}{2} \right) \sqrt{(l \pm m)(l \mp m + 1)}, \quad (32)$$

$$I_{0,\pm 2}(l, m) = -\frac{l^2 + l - 3}{(2l-1)(2l+3)} \sqrt{(l \pm m)(l \pm m - 1)(l \mp m + 1)(l \mp m + 2)}. \quad (33)$$

For an arbitrarily oriented magnetic field, even when $l = l'$, non-zero $I_{0,\pm 1}$ and $I_{0,\pm 2}$ indicate that there are non-zero non-equal m, m' correlations. The coefficients $I_{0,\pm a}(m)$ have the following symmetries,

$$I_{0,\pm a}(m) = (-1)^a I_{0,\mp a}(-m) = I_{0,\mp a}(m \mp a) = I_{0,\pm a}(-(m \pm a)), \quad (34)$$

where $a = |m - m'|$ and thus takes values 0, 1, and 2.

Taking the complex conjugate of $C_l^{(m,m')}$ it is straightforward to see that

$$C_l^{(m,m')*}(\Theta_B, \phi_B) = C_l^{(m',m)}(\Theta_B, \phi_B) = C_l^{(m,m')}(\Theta_B, -\phi_B), \quad (35)$$

so exchanging m and m' corresponds to replacing ϕ_B by $-\phi_B$, and effectively corresponds to complex conjugation.

The imaginary part of $C_l^{(m,m')}$ is

$$\begin{aligned} A_{m,m'}^{l=l'}(\Theta_B, \phi_B) &= -\frac{i}{2} \left\{ C_l^{(m,m')} - C_l^{(m,m')*} \right\} = \text{Im} \left(\langle a_{lm}^* a_{lm'} \rangle \right) \\ &= -\sin \Theta_B \sin \phi_B \left\{ 2 \cos \Theta_B \left[\delta_{m,m'-1} I_{0,-1}(m) - \delta_{m,m'+1} I_{0,+1}(m) \right] \right. \\ &\quad \left. + \sin \Theta_B \cos \phi_B \left[\delta_{m,m'-2} I_{0,-2}(m) - \delta_{m,m'+2} I_{0,+2}(m) \right] \right\} I_d^{l,l}. \end{aligned} \quad (36)$$

So a measured imaginary part of $\langle a_{lm}^* a_{lm'} \rangle$ will indicate the direction of the magnetic field in space. For a magnetic field along \mathbf{z} the imaginary part vanishes. The imaginary part also vanishes when $\phi_B = 0$. These imply that $A_{m,m'}^{l=l'}(\Theta_B, \phi_B) \propto |\mathbf{b} \times \hat{\mathbf{z}}|$.

B. $l = l' \pm 2$ correlations

Making use of the symmetries, we need to determine

$$\langle a_{l-1,m}^* a_{l+1,m'} \rangle = D_{l-1,l+1}^{(m,m')}(\Theta_B, \phi_B), \quad \langle a_{l+1,m}^* a_{l-1,m'} \rangle = D_{l+1,l-1}^{(m,m')}(\Theta_B, \phi_B). \quad (37)$$

Proceeding in a similar way as for the $l = l'$ case we find

$$\begin{aligned} D_{l\mp 1, l\pm 1}^{(m,m')}(\Theta_B, \phi_B) &= \left\{ (3 \cos^2 \Theta_B - 1) \delta_{mm'} I_{\pm 2,0} \right. \\ &\quad \left. + 2 \sin \Theta_B \cos \Theta_B \left[e^{-i\phi_B} \delta_{m,m'-1} I_{\pm 2,-1} + e^{i\phi_B} \delta_{m,m'+1} I_{\pm 2,+1} \right] \right. \\ &\quad \left. + \frac{1}{2} \sin^2 \Theta_B \left[e^{-2i\phi_B} \delta_{m,m'-2} I_{\pm 2,-2} + e^{2i\phi_B} \delta_{m,m'+2} I_{\pm 2,+2} \right] \right\} I_d^{(l\mp 1, l\pm 1)}. \end{aligned} \quad (38)$$

Defining the coefficient

$$\mathcal{I}(l) = \frac{(l+2)(l-1)}{2(2l+1)\sqrt{(2l-1)(2l+3)}}, \quad (39)$$

we list and discuss separately the $m = m'$ ($I_{\pm 2,0}$), $m = m' \pm 1$ ($I_{\pm 2,\pm 1}$), and $m = m' \pm 2$ ($I_{\pm 2,\pm 2}$) term coefficients of Eq. (38) in what follows.

For the $m = m'$ term we find

$$I_{\pm 2,0}(l, m) = -\sqrt{(l+m)(l-m)(l-m+1)(l+m+1)} \mathcal{I}, \quad (40)$$

which results in

$$D_{l-1,l+1}^{(m=m')} = \langle a_{l-1,m}^* a_{l+1,m} \rangle = \langle a_{l+1,m}^* a_{l-1,m} \rangle = D_{l+1,l-1}^{(m=m')} = (3 \cos^2 \Theta_B - 1) I_{+2,0} I_d^{(l-1,l+1)}, \quad (41)$$

where $I_d^{(l,l')}$ is defined in Eq. (24). From Eq. (41), when $\Theta_B = 0$ the part of $\langle a_{lm}^* a_{l'm'} \rangle$ proportional to $\delta_{mm'}$ is $2I_{+2,0} I_d^{(l-1,l+1)}$, which coincides with the result of Ref. [17].

The part of the right hand side of Eq. (38) proportional to $\sin \Theta_B \cos \Theta_B$ contains terms proportional to $\delta_{l \mp 1, l \pm 1}$ and $\delta_{m, m'-1}$ or $\delta_{m, m'+1}$. The coefficients in these terms are

$$I_{+2,\mp 1}(l, m) = \sqrt{(l \mp m)(l \pm m)(l \pm m + 1)(l \pm m + 2)} \mathcal{I} = \sqrt{\frac{(l \mp m)!}{(l \mp m - 1)!} \frac{(l \pm m + 2)!}{(l \pm m - 1)!}} \mathcal{I}, \quad (42)$$

and

$$I_{-2,\mp 1}(l, m) = -\sqrt{(l \pm m + 1)(l \mp m + 1)(l \mp m)(l \mp m - 1)} \mathcal{I} = -\sqrt{\frac{(l \pm m + 1)!}{(l \pm m)!} \frac{(l \mp m + 1)!}{(l \mp m - 2)!}} \mathcal{I}. \quad (43)$$

These have the following symmetries,

$$I_{\pm 2,\pm 1}(m) = -I_{\pm 2,\mp 1}(-m) = I_{\mp 2,\mp 1}(m \mp 1) = -I_{\mp 2,\pm 1}(-(m \mp 1)), \quad (44)$$

i.e., the cross correlations between $l - 1$ and $l + 1$ multipole coefficients are the negative of those between $l + 1$ and $l - 1$ multipole coefficients provided m is replaced by $-(m \pm 1)$.

The coefficients of the last set of terms in Eq. (38) with $m = m' \pm 2$ are

$$I_{+2,\mp 2}(l, m) = -\sqrt{(l \pm m)(l \pm m + 1)(l \pm m + 2)(l \pm m + 3)} \mathcal{I} = -\sqrt{\frac{(l \pm m + 3)!}{(l \pm m - 1)!}} \mathcal{I}, \quad (45)$$

and

$$I_{-2,\mp 2}(l, m) = -\sqrt{(l \mp m + 1)(l \mp m)(l \mp m - 1)(l \mp m - 2)} \mathcal{I} = -\sqrt{\frac{(l \mp m + 1)!}{(l \mp m - 3)!}} \mathcal{I}. \quad (46)$$

These have the following symmetries,

$$I_{\pm 2,\pm 2}(m) = I_{\pm 2,\mp 2}(-m) = I_{\mp 2,\mp 2}(m \mp 2) = I_{\mp 2,\pm 2}(-(m \mp 2)), \quad (47)$$

i.e., the correlations between $l - 1$ and $l + 1$ multipole coefficients are the negative of those between $l + 1$ and $l - 1$ multipole coefficients provided m is replaced by $-(m \pm 2)$.

Equations (40), (44), and (47) can be combined into one set of equations that reflect the symmetry of the $I_{\pm 2,\pm a}$ ($a = 0, 1, 2$) coefficients, similar to Eq. (34), resulting in one set of equations for both the $l = l'$ and $l = l' \pm 2$ cases,

$$I_{\pm b,\pm a}(m) = (-1)^a I_{\pm b,\mp a}(-m) = I_{\mp b,\mp a}(m \mp a) = I_{\mp b,\pm a}(-(m \mp a)), \quad (48)$$

where $b = 0$ or 2 . On the other hand, the magnitude of the cross-correlation coefficients for $l = l'$ and $l = l' \pm 2$ are different; while all terms for the $l = l' \pm 2$ case are proportional to \mathcal{I} , this is not true for the $l = l'$ coefficients.

The non-zero off-diagonal correlations power spectrum terms $D_{l\mp 1, l\pm 1}^{(m, m')}(\Theta_B, \phi_B)$ are given by Eq. (38). Taking the complex conjugate we see

$$D_{l\mp 1, l\pm 1}^{(m, m')\star}(\Theta_B, \phi_B) = D_{l\mp 1, l\pm 1}^{(m', m)}(\Theta_B, \phi_B) = D_{l\mp 1, l\pm 1}^{(m, m')}(\Theta_B, -\phi_B), \quad (49)$$

so, as for the $C_l^{(m, m')}$ function in Eq. (35), complex conjugation is equivalent to exchanging ϕ_B and $-\phi_B$. Consequently, the $D_{l\mp 1, l\pm 1}^{(m, m')}(\Theta_B, \phi_B)$ are complex functions, with imaginary part

$$\begin{aligned} -\frac{i}{2} \left\{ D_{l\mp 1, l\pm 1}^{(mm')} - D_{l\mp 1, l\pm 1}^{(m, m')\star} \right\} &= \text{Im} \left(\langle a_{l\mp 1, m}^* a_{l\pm 1, m'} \rangle \right) \\ &= -\sin \Theta_B \sin \phi_B \left\{ 2 \cos \Theta_B \left[\delta_{m, m'-1} I_{\pm 2, -1}(m) - \delta_{m, m'+1} I_{\pm 2, -1}(-m) \right] \right. \\ &\quad \left. + \sin \Theta_B \cos \phi_B \left[\delta_{m, m'-2} I_{\pm 2, -2}(m) - \delta_{m, m'+2} I_{\pm 2, -2}(-m) \right] \right\} I_d^{(l\mp 1, l\pm 1)}. \end{aligned} \quad (50)$$

So non-zero correlations between non-equal m multipole numbers result in an imaginary (antisymmetric) part of $\langle a_{l\mp 1, m}^* a_{l\pm 1, m'} \rangle$ which effectively breaks the symmetry between the north and south hemispheres.

IV. TEMPERATURE CORRELATIONS

In this Section we derive the CMB temperature fluctuation two-point correlation function $\langle \Delta T/T(\mathbf{n}) \Delta T/T(\mathbf{n}') \rangle$ that is induced purely from the homogeneous magnetic field. This must be added to the usual CMB temperature two-point correlation function. Since the Alfvén velocity is small this magnetic field induced CMB anisotropy is a small correction to the “primary” CMB temperature fluctuations. On the other hand, the effects that we discuss here vanish in the standard cosmological model so a non-zero correlation between l and $l \pm 2$ multipole coefficients might indicate the presence of an homogeneous cosmological magnetic field.

The two-point temperature correlation function can be written as

$$\begin{aligned} \left\langle \frac{\Delta T}{T}(\mathbf{n}) \frac{\Delta T}{T}(\mathbf{n}') \right\rangle &= \frac{1}{2} \sum_{l, l'} \sum_{m, m'} \left[\langle a_{lm}^* a_{l'm'} \rangle Y_{lm}^*(\mathbf{n}) Y_{l'm'}(\mathbf{n}') + \langle a_{lm} a_{l', m'}^* \rangle Y_{lm}(\mathbf{n}) Y_{l'm'}^*(\mathbf{n}') \right] \\ &= \left\langle \frac{\Delta T}{T}(\mathbf{n}) \frac{\Delta T}{T}(\mathbf{n}') \right\rangle \Big|^{l=l'} + \left\langle \frac{\Delta T}{T}(\mathbf{n}) \frac{\Delta T}{T}(\mathbf{n}') \right\rangle \Big|^{l=l' \pm 2}, \end{aligned} \quad (51)$$

where we introduce complex conjugation to symmetrize over \mathbf{n} and \mathbf{n}' . From Eqs. (30) and (38) we see that both terms in the correlation function (the contribution that are diagonal in l as well as those that are between l and $l \pm 2$ multipole coefficients) contain three kinds of terms, those proportional to: (1) $3 \cos^2 \Theta_B - 1$; (2) $\sin \Theta_B \cos \Theta_B e^{\pm i \phi_B}$; and, (3) $\sin^2 \Theta_B e^{\pm 2i \phi_B}$. We derive the contributions from these terms in App. B.

Using the results of Sec. III for the multipole coefficients, and the addition theorem of Eq. (A24), we find, from App. B, the diagonal $l = l'$ correlation contribution,

$$\begin{aligned} \left\langle \frac{\Delta T}{T}(\mathbf{n}) \frac{\Delta T}{T}(\mathbf{n}') \right\rangle \Big|^{l=l'} &= \frac{1}{4\pi} \sum_l \frac{l(l+1)(2l+1)}{(2l-1)(2l+3)} \left\{ (2l^2 + 2l - 3) P_l \right. \\ &\quad \left. + 2(l^2 + l - 3) \left[(\mathbf{b} \cdot \mathbf{n})(\mathbf{b} \cdot \mathbf{n}') [2P_{l-1}'' + (2l-1)P_l'] - [(\mathbf{b} \cdot \mathbf{n})^2 + (\mathbf{b} \cdot \mathbf{n}')^2] P_l'' + P_{l-1}' - l^2 P_l \right] \right\} I_d^{(l, l)}, \end{aligned} \quad (52)$$

and for the off-diagonal $l = l' \pm 2$ correlation contribution, where we use the addition theorem of Eq. (A23), we find, from App. B,

$$\begin{aligned} \left\langle \frac{\Delta T}{T}(\mathbf{n}) \frac{\Delta T}{T}(\mathbf{n}') \right\rangle \Big|^{l=l' \pm 2} &= \\ \frac{1}{4\pi} \sum_l \frac{2(l+2)(l-1)}{2l+1} \left\{ 2(\mathbf{b} \cdot \mathbf{n})(\mathbf{b} \cdot \mathbf{n}') P_l'' - \frac{1}{2} \left[(\mathbf{b} \cdot \mathbf{n})^2 + (\mathbf{b} \cdot \mathbf{n}')^2 \right] \left[3P_l'(x) + 2(\mathbf{n} \cdot \mathbf{n}') P_l'' \right] + P_l' \right\} I_d^{(l-1, l+1)}, \end{aligned} \quad (53)$$

where the argument of the Legendre polynomials and derivatives in Eqs. (52) and (53) are $\mathbf{n} \cdot \mathbf{n}'$. If \mathbf{b} is perpendicular to \mathbf{n} or \mathbf{n}' , or if $\mathbf{n} = \mathbf{n}'$, the above expressions simplify considerably.

To obtain the CMB temperature anisotropy two-point correlation function, Eqs. (52) and (53), in terms of the initial vorticity spectrum $P_{\Omega_0} = P_0 k^{n_\Omega} / k_S^{n_\Omega+3}$, the integrals $I_d^{(l, l)}$ and $I_d^{(l-1, l+1)}$, Eq. (24), must be evaluated. These

can be evaluated using an analytical approximations, for details see App. A.3 and the Appendix of Ref. [27]. The result depends sensitively on the initial vorticity perturbation spectral index (n_Ω), Eq. (10).

Accounting for the solution of Eq. (8), the symmetric part P_Ω of the resulting vorticity perturbation spectrum is characterized by the spectral index $n_\Omega + 2$, i.e., $P_\Omega \propto k^{n_\Omega+2}$, while the perturbed magnetic field \mathbf{B}_1 inherits the initial vorticity spectral index n_Ω . To avoid a divergence of the energy density spectrum E_Ω of the resulting vorticity perturbations on super-Hubble-radius scales, we require $n_\Omega \geq -7$ ($E_\Omega(k) \propto k^{(n_\Omega+4)}$ and the three-dimensional wavenumber integration gives an additional factor of k^3). Requiring a non-divergent temperature two-point correlation function at large wavenumbers leads to $n_\Omega \leq -1$ [17]. Another important value of n_Ω follows from the requirement that the initial vorticity field energy density not diverge at small wavenumbers, which results in $n_\Omega \geq -5$. Requiring that the inequality $|\mathbf{\Omega}_0|^2 k^3 \leq v_A^2$ (resulting from $B_1 \leq B_0$) [17] hold on any scale inside the Hubble radius at decoupling, i.e., for $k \geq 1/t_{\text{dec}}$, we need [17]

$$2P_0 \left(\frac{k}{k_S} \right)^{n_\Omega+3} \leq v_A^2, \quad (54)$$

which implies (accounting for $k \leq k_S$) $2P_0 \leq v_A^2$ for $n_\Omega \geq -3$. As shown in Ref. [17] this inequality leads to an unconstrained magnetic field for $n_\Omega \geq -3$. Since the more interesting results are in the range $n_\Omega \in (-7, -3)$, we adopt here -3 as the upper value for n_Ω . In this range of the spectral index n_Ω the integral can be accurately computed analytically. When $n_\Omega \geq -1$ the integral can be computed reasonable accurately in the analytic approximation [27].

Using Eq. (A25), for $n_\Omega \in (-7, -1)$, we find,

$$I_d^{(l,l)} = \frac{P_0 v_A^2 \eta_{\text{dec}}^2 \Gamma(-n_\Omega/2 - 1/2)}{2\sqrt{\pi} (k_S \eta_0)^{n_\Omega+3} \eta_0^2 \Gamma(-n_\Omega/2)} \frac{\Gamma(l + 3/2 + n_\Omega/2)}{\Gamma(l + 1/2 - n_\Omega/2)}, \quad (55)$$

$$I_d^{(l-1,l+1)} = \frac{P_0 v_A^2 \eta_{\text{dec}}^2 (n_\Omega + 2) \Gamma(-n_\Omega/2 - 1/2)}{2\sqrt{\pi} (k_S \eta_0)^{n_\Omega+3} \eta_0^2 n_\Omega \Gamma(-n_\Omega/2)} \frac{\Gamma(l + 3/2 + n_\Omega/2)}{\Gamma(l + 1/2 - n_\Omega/2)}. \quad (56)$$

When $n_\Omega \geq -7$ the quadrupole ($l = 2$) moment does not diverge, see the last term on the right hand sides $\propto \Gamma(l + 3/2 + n_\Omega/2)$. For large enough l 's this last term is $\propto l^{n_\Omega+1}$ and makes both integrals decay (for $n_\Omega \leq -1$) with l as $l^{n_\Omega+1}$ for increasing l .

V. CONCLUSIONS

We derive the CMB temperature anisotropy two-point correlation function sourced by vorticity perturbations induced by an homogeneous magnetic field. We extend the analysis of Ref. [17] by considering a magnetic field that is arbitrarily oriented with respect to the galactic plane. We consider a weak magnetic field, and since it is uniform and points in a fixed direction it breaks spatial isotropy. In this case the only non-zero correlations between multipole coefficients are between those that have $\Delta l = 0$ and $\Delta l = \pm 2$, and $\Delta m = 0$, $\Delta m = \pm 1$, and $\Delta m = \pm 2$, and we have accounted for all non-zero correlations. Even though we have computed only the two-point correlation function, such off-diagonal correlations indicate that in this model the CMB temperature anisotropy is non-gaussian [31]. Such an homogeneous magnetic field might explain the tentative large-scale non-gaussianity of the CMB temperature anisotropy (also see Refs. [14, 15, 32]). Our results, when used in analyses of the WMAP data, as well as anticipated PLANCK data, could be used to search for or limit an homogeneous cosmological magnetic field. The off-diagonal correlations we have found might be a unique signature of such a field.

While our results were obtained assuming an homogeneous magnetic field, they can be extended to an almost homogeneous cosmological magnetic field with correlation length larger than the Hubble radius today. Such a field, with a large enough amplitude, can be generated by quantum-mechanical zero-point fluctuations during inflation. In this case the spectral index of the magnetic field is around $n_B = -3$. See Ref. [16] and the more recent studies in Refs. [33]. Limits on a cosmological magnetic field that can be obtained through the formalism we have developed here will compliment those obtained through the CMB polarization Faraday rotation effect [30, 34, 35, 36] and the non-zero cross-correlations between CMB temperature and B -polarization anisotropies [20, 37].

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APPENDIX A: USEFUL MATHEMATICAL FORMULAE

In this Appendix we list various mathematical results we use in the computations.

1. Spherical harmonics and Legendre polynomials

The orthonormality relation for spherical harmonics is

$$\int d\Omega_{\hat{\mathbf{k}}} Y_{rq}^*(\hat{\mathbf{k}}) Y_{r'q'}(\hat{\mathbf{k}}) = \delta_{rr'} \delta_{qq'} . \quad (\text{A1})$$

The recurrence relations for spherical harmonics are [38]

$$\cos \theta Y_{lm}(\theta, \phi) = \alpha_{l+1,m}^{(0)} Y_{l+1,m}(\theta, \phi) + \beta_{l-1,m}^{(0)} Y_{l-1,m}(\theta, \phi) , \quad (\text{A2})$$

$$\sin \theta e^{\pm i\phi} Y_{lm}(\theta, \phi) = \alpha_{l+1,m\pm 1}^{(\pm)} Y_{l+1,m\pm 1}(\theta, \phi) + \beta_{l-1,m\pm 1}^{(\pm)} Y_{l-1,m\pm 1}(\theta, \phi) , \quad (\text{A3})$$

where

$$\alpha_{l,m}^{(0)} = \sqrt{\frac{(l-m)(l+m)}{(2l-1)(2l+1)}} , \quad \alpha_{l,m}^{(\pm)} = \mp \sqrt{\frac{(l\pm m-1)(l\pm m)}{(2l-1)(2l+1)}} , \quad (\text{A4})$$

$$\beta_{l,m}^{(0)} = \sqrt{\frac{(l-m+1)(l+m+1)}{(2l+1)(2l+3)}} , \quad \beta_{l,m}^{(\pm)} = \pm \sqrt{\frac{(l\mp m+2)(l\mp m+1)}{(2l+1)(2l+3)}} . \quad (\text{A5})$$

Legendre polynomials of order l are defined by the sum

$$P_l(\mathbf{n} \cdot \mathbf{n}') = \frac{4\pi}{2l+1} \sum_{m=-l}^l Y_{lm}^*(\mathbf{n}) Y_{lm}(\mathbf{n}') . \quad (\text{A6})$$

Equations (A1) and (A6) imply

$$\int d\Omega_{\hat{\mathbf{q}}} P_i(\mathbf{n} \cdot \hat{\mathbf{q}}) P_j(\mathbf{n}' \cdot \hat{\mathbf{q}}) = \frac{4\pi}{2j+1} \delta_{ij} P_j(\mathbf{n} \cdot \mathbf{n}') . \quad (\text{A7})$$

2. Vector spherical harmonics

a. Vector spherical harmonics components

The $\mathbf{Y}_{lm}^{(\lambda)}(\mathbf{n})$ ($\lambda = -1, 0, +1$) vector spherical harmonics are (Eqs. (6,7), p. 210, [29])

$$\begin{aligned} \mathbf{Y}_{lm}^{(+1)}(\mathbf{n}) &= \frac{1}{\sqrt{l(l+1)}} \nabla_{\Omega} Y_{lm}(\mathbf{n}), \\ \mathbf{Y}_{lm}^{(0)}(\mathbf{n}) &= \frac{-i}{\sqrt{l(l+1)}} [\mathbf{n} \times \nabla_{\Omega}] Y_{lm}(\mathbf{n}), \\ \mathbf{Y}_{lm}^{(-1)}(\mathbf{n}) &= \mathbf{n} Y_{lm}(\mathbf{n}), \end{aligned} \quad (\text{A8})$$

where ∇_{Ω} denotes the angular part of the ∇ operator. The $\mathbf{Y}_{lm}^{(\lambda)}(\mathbf{n})$ vector spherical harmonics are related to the $\mathbf{Y}_{lm}^j(\mathbf{n})$ vector spherical harmonics through (Eq. (9), p. 210, [29])

$$\mathbf{Y}_{rq}^{(\pm 1)}(\mathbf{n}) = \sqrt{\frac{r}{2r+1}} \mathbf{Y}_{rq}^{r\pm 1}(\mathbf{n}) \pm \sqrt{\frac{r\pm 1}{2r+1}} \mathbf{Y}_{rq}^{r\mp 1}(\mathbf{n}) , \quad \mathbf{Y}_{rq}^{(0)}(\mathbf{n}) = \mathbf{Y}_{rq}^r . \quad (\text{A9})$$

The $\mathbf{Y}_{lm}^j(\mathbf{n})$ vector spherical harmonics are related to the usual $Y_{lm}(\mathbf{n})$ spherical harmonics through (Eqs. (9), (11-13), pp. 210-211, [29])

$$\mathbf{n}Y_{lm}(\mathbf{n}) = \sqrt{\frac{l}{2l+1}}\mathbf{Y}_{lm}^{l-1}(\mathbf{n}) - \sqrt{\frac{l+1}{2l+1}}\mathbf{Y}_{lm}^{l+1}(\mathbf{n}), \quad (\text{A10})$$

$$\mathbf{Y}_{lm}^j(\mathbf{n}) = \sum_{s=-1}^1 |\mathbf{Y}_{lm}^j(\mathbf{n})|^s \mathbf{e}_s = \sum_{s=-1}^1 (-1)^s |\mathbf{Y}_{lm}^j(\mathbf{n})|_{-s} \mathbf{e}_s, \quad (\text{A11})$$

$$|\mathbf{Y}_{lm}^j(\mathbf{n})|^s = (-1)^s |\mathbf{Y}_{lm}^j(\mathbf{n})|_{-s} = C_{j,m-s,1,s}^{lm} Y_{j,m-s}(\mathbf{n}), \quad (\text{A12})$$

where $|\mathbf{Y}_{lm}^j(\mathbf{n})|^s$ and $|\mathbf{Y}_{lm}^j(\mathbf{n})|_s$ are contravariant and covariant components, $\mathbf{e}_s (s = \pm 1, 0)$ are unit covariant vectors, and $C_{j,m-s,1,s}^{lm}$ are Clebsch-Gordon coefficients related to the $\alpha_{l,m}^{(\pm)}$ and $\beta_{l,m}^{(\pm)}$ coefficients of Eqs. (A4)–(A5).

The contravariant components of the $\mathbf{Y}_{lm}^j(\mathbf{n})$ vector spherical harmonics are related to the usual spherical harmonics through (pp. 211-212, [29])

$$\begin{aligned} |\mathbf{Y}_{rq}^{r+1}(\mathbf{n})|^{(\pm 1)} &= \sqrt{\frac{(r \mp q + 1)(r \mp q + 2)}{2(r+1)(2r+3)}} Y_{r+1,q \mp 1}(\mathbf{n}), & |\mathbf{Y}_{rq}^{r+1}(\mathbf{n})|^{(0)} &= -\sqrt{\frac{(r-q+1)(r+q+1)}{(r+1)(2r+3)}} Y_{r+1,q}(\mathbf{n}), \\ |\mathbf{Y}_{rq}^r(\mathbf{n})|^{(\pm 1)} &= \mp \sqrt{\frac{(r \pm q)(r \mp q + 1)}{2r(r+1)}} Y_{r,q \mp 1}(\mathbf{n}), & |\mathbf{Y}_{rq}^r(\mathbf{n})|^{(0)} &= \frac{q}{\sqrt{r(r+1)}} Y_{r,q}(\mathbf{n}), \\ |\mathbf{Y}_{rq}^{r-1}(\mathbf{n})|^{(\pm 1)} &= \sqrt{\frac{(r \pm q)(r \pm q - 1)}{2r(2r-1)}} Y_{r-1,q \mp 1}(\mathbf{n}), & |\mathbf{Y}_{rq}^{r-1}(\mathbf{n})|^{(0)} &= \sqrt{\frac{(r-q)(r+q)}{r(2r-1)}} Y_{r-1,q}(\mathbf{n}). \end{aligned} \quad (\text{A13})$$

b. Vector plane wave expansion

A vector plane wave field can be expanded in vector spherical harmonics (Eq. (132), p. 228, [29]) as

$$\mathbf{v}(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{n}t} = \sum_{l,\lambda,m} A_{lm}^{(\lambda)} \mathbf{Y}_{lm}^{(\lambda)}(\mathbf{n}), \quad (\text{A14})$$

where $\lambda = -1, 0, 1$, and the expansion coefficients for a transverse field $\mathbf{v}(\mathbf{k})$ ($\mathbf{v}(\mathbf{k}) \cdot \mathbf{k} = 0$) are

$$A_{lm}^{(-1)} = 4\pi i^{l-1} \sqrt{l(l+1)} \frac{j_l(kt)}{kt} \mathbf{v}(\mathbf{k}) \cdot \mathbf{Y}_{lm}^{(+1)\star}(\hat{\mathbf{k}}), \quad (\text{A15})$$

$$A_{lm}^{(0)} = 4\pi i^l j_l(kt) \mathbf{v}(\mathbf{k}) \cdot \mathbf{Y}_{lm}^{(0)\star}(\hat{\mathbf{k}}), \quad (\text{A16})$$

$$A_{lm}^{(+1)} = -4\pi i^{l+1} \left(\frac{j_l(kt)}{kt} + j'_l(kt) \right) \mathbf{v}(\mathbf{k}) \cdot \mathbf{Y}_{lm}^{(+1)\star}(\hat{\mathbf{k}}). \quad (\text{A17})$$

The terms $\propto \mathbf{v}(\mathbf{k}) \cdot \mathbf{Y}_{lm}^{(-1)\star}(\hat{\mathbf{k}})$ in the $A_{lm}^{(\pm 1)}$ coefficients vanish because $\mathbf{v}(\hat{\mathbf{k}}) \cdot \mathbf{Y}_{lm}^{(-1)\star}(\hat{\mathbf{k}}) = 0$ as a consequence of $\hat{\mathbf{k}} \cdot \mathbf{v}(\mathbf{k}) = 0$.

c. Decomposition of vector spherical harmonics

In the helicity basis where the angles Θ and ϕ are defined by the unit wavevector $\hat{\mathbf{k}}$, vector spherical harmonics are given by (Eq. (35), p. 215, [29])

$$\begin{aligned} \mathbf{Y}_{lm}^{(+1)}(\hat{\mathbf{k}}) &= \sqrt{\frac{2l+1}{8\pi}} \left[D_{-1,-m}^l(0, \Theta, \phi) \mathbf{e}'_{+1}(\hat{\mathbf{k}}) + D_{1,-m}^l(0, \Theta, \phi) \mathbf{e}'_{-1}(\hat{\mathbf{k}}) \right], \\ \mathbf{Y}_{lm}^{(0)}(\hat{\mathbf{k}}) &= \sqrt{\frac{2l+1}{8\pi}} \left[-D_{-1,-m}^l(0, \Theta, \phi) \mathbf{e}'_{+1}(\hat{\mathbf{k}}) + D_{1,-m}^l(0, \Theta, \phi) \mathbf{e}'_{-1}(\hat{\mathbf{k}}) \right], \\ \mathbf{Y}_{lm}^{(-1)}(\hat{\mathbf{k}}) &= \sqrt{\frac{2l+1}{4\pi}} D_{0,-m}^l(0, \Theta, \phi) \mathbf{e}'_0(\hat{\mathbf{k}}). \end{aligned} \quad (\text{A18})$$

Here the helicity basis vectors \mathbf{e}_μ' are defined above Eq. (21) and the Wigner D functions are defined (Eq. (1), p. 76, [29]) by

$$D_{m,m'}^l(\alpha, \beta, \gamma) = e^{-im\alpha} d_{mm'}^l(\beta) e^{-im'\gamma}, \quad (\text{A19})$$

where $d_{m,m'}^l(\beta)$ is a real function defined in Sec. 4.3 of Ref. [29].

d. Addition theorems for and sums of vector spherical harmonics

We have need for the following sums of vector spherical harmonics $\mathbf{Y}_{rq}^{(\lambda)}(\mathbf{n})$ (Eqs. (80), p. 221, [29])

$$4\pi \sum_{q=-r}^r Y_{rq}^*(\mathbf{n}) \mathbf{Y}_{rq}^{(-1)}(\mathbf{n}) = (2r+1)\mathbf{n}, \quad 4\pi \sum_{q=-r}^r Y_{rq}^*(\mathbf{n}) \mathbf{Y}_{rq}^{(0)}(\mathbf{n}) = 4\pi \sum_{q=-r}^r Y_{rq}^*(\mathbf{n}) \mathbf{Y}_{rq}^{(1)}(\mathbf{n}) = 0. \quad (\text{A20})$$

Some addition theorems for \mathbf{Y}_{rq}^R are (p. 223, [29]),

$$4\pi \sum_{q=-r}^r \mathbf{Y}_{rq}^{R*}(\mathbf{n}_1) \cdot \mathbf{Y}_{rq}^{R'}(\mathbf{n}_2) = \delta_{RR'} (2r+1) P_R(\mathbf{n}_1 \cdot \mathbf{n}_2), \quad (\text{A21})$$

and

$$4\pi \sum_{q=-r}^r \mathbf{Y}_{rq}^{r\pm 1*}(\mathbf{n}_1) \times \mathbf{Y}_{rq}^{r\mp 1}(\mathbf{n}_2) = 0. \quad (\text{A22})$$

The most general form of the addition theorems for vector spherical harmonics are given in Sect. 7.3.11 of Ref. [29]. Here we list two for arbitrary real vectors \mathbf{a}_1 and \mathbf{a}_2 .

$$4\pi \sum_{q=-r}^r (\mathbf{a}_1 \cdot \mathbf{Y}_{rq}^{r\pm 1*}(\mathbf{n}_1)) (\mathbf{a}_2 \cdot \mathbf{Y}_{rq}^{r\mp 1}(\mathbf{n}_2)) = \frac{1}{\sqrt{r(r+1)}} \left\{ [(\mathbf{a}_1 \cdot \mathbf{n}_1)(\mathbf{a}_2 \cdot \mathbf{n}_2) + (\mathbf{a}_1 \cdot \mathbf{n}_2)(\mathbf{a}_2 \cdot \mathbf{n}_1)] P_r'' \right. \\ \left. - (\mathbf{a}_1 \cdot \mathbf{n}_1)(\mathbf{a}_2 \cdot \mathbf{n}_1) P_{r\pm 1}'' - (\mathbf{a}_1 \cdot \mathbf{n}_2)(\mathbf{a}_2 \cdot \mathbf{n}_2) P_{r\mp 1}'' + (\mathbf{a}_1 \cdot \mathbf{a}_2) P_r' \right\}, \quad (\text{A23})$$

$$4\pi \sum_{q=-r}^r (\mathbf{a}_1 \cdot \mathbf{Y}_{rq}^{r*}(\mathbf{n}_1)) (\mathbf{a}_2 \cdot \mathbf{Y}_{rq}^r(\mathbf{n}_2)) = \frac{2r+1}{r(r+1)} \left\{ -(\mathbf{a}_1 \cdot \mathbf{n}_1)(\mathbf{a}_2 \cdot \mathbf{n}_2) [P_{r-1}'' + (r-1)P_r'] \right. \\ \left. - (\mathbf{a}_1 \cdot \mathbf{n}_2)(\mathbf{a}_2 \cdot \mathbf{n}_1) [P_{r-1}'' + rP_r'] + [(\mathbf{a}_1 \cdot \mathbf{n}_1)(\mathbf{a}_2 \cdot \mathbf{n}_1) + (\mathbf{a}_1 \cdot \mathbf{n}_2)(\mathbf{a}_2 \cdot \mathbf{n}_2)] P_r'' + (\mathbf{a}_1 \cdot \mathbf{a}_2) [r^2 P_r - P_{r-1}''] \right\}. \quad (\text{A24})$$

In these expressions P_r' and P_r'' are derivatives of Legendre polynomials and we have omitted the arguments of Legendre polynomials and their derivatives, abbreviating $P_r(\mathbf{n}_1 \cdot \mathbf{n}_2)$ as P_r , etc.

3. Integrals of spherical Bessel functions

Here we present an analytical approximate formula to compute the integral $I_d^{(l,l')}$ of Eq. (24). The integrals that we need to evaluate are of the form $\int_0^{x_S} dx J_p(ax) J_q(ax) x^{-b}$, which contain products of Bessel functions. For $b > 0$ when the integral converges and is dominated by $x \ll x_S$, the upper limit x_S can be replaced by ∞ (with an accuracy of a few percent for $b > 1$, and 15–30 % for $0 < b < 1$, depending on the value of $p - q$). We can then use Eq. (6.574.2) of Ref. [39],

$$\int_0^\infty dx J_p(ax) J_q(ax) x^{-b} = \frac{a^{b-1} \Gamma(b) \Gamma((p+q-b+1)/2)}{2^b \Gamma((-p+q+b+1)/2) \Gamma((p+q+b+1)/2) \Gamma((p-q+b+1)/2)}, \quad (\text{A25})$$

which is valid for $\text{Re}(p+q+1) > \text{Re } b > 0$ and $a > 0$.

APPENDIX B: COMPUTATION OF TEMPERATURE CORRELATION FUNCTIONS

The diagonal and off-diagonal correlation parts of the temperature anisotropy two-point correlation function of Eq. (51) are

$$\left\langle \frac{\Delta T}{T}(\mathbf{n}) \frac{\Delta T}{T}(\mathbf{n}') \right\rangle \Big|_{l=l'}^{l=l'} = \frac{1}{2} \sum_l \sum_{m,m'} \left[C_l^{(m,m')} Y_{l,m}^*(\mathbf{n}) Y_{l,m'}(\mathbf{n}') + C_l^{(m,m')*} Y_{l,m}(\mathbf{n}) Y_{l,m'}^*(\mathbf{n}') \right], \quad (\text{B1})$$

and

$$\begin{aligned} \left\langle \frac{\Delta T}{T}(\mathbf{n}) \frac{\Delta T}{T}(\mathbf{n}') \right\rangle \Big|_{l=l' \pm 2}^{l=l' \pm 2} = \frac{1}{2} \sum_l \sum_{m,m'} & \left[D_{l-1,l+1}^{(m,m')} Y_{l-1,m}^*(\mathbf{n}) Y_{l+1,m'}(\mathbf{n}') + D_{l+1,l-1}^{(m,m')} Y_{l+1,m}^*(\mathbf{n}) Y_{l-1,m'}(\mathbf{n}') \right. \\ & \left. + D_{l-1,l+1}^{(m,m')*} Y_{l-1,m}(\mathbf{n}) Y_{l+1,m'}^*(\mathbf{n}') + D_{l+1,l-1}^{(m,m')*} Y_{l+1,m}(\mathbf{n}) Y_{l-1,m'}^*(\mathbf{n}') \right]. \quad (\text{B2}) \end{aligned}$$

In this Appendix we summarize the results of a computation of these terms.

We first compute the diagonal $l = l'$ correlations of Eq. (B1). From Eq. (30) we see that there are three different types of terms, which we now list. The first type of term is the $l = l'$ and $m = m'$ correlation proportional to $3 \cos^2 \Theta_B - 1$ on the right hand side of Eq. (30), which results in

$$\begin{aligned} 4\pi \left\langle \frac{\Delta T}{T}(\mathbf{n}) \frac{\Delta T}{T}(\mathbf{n}') \right\rangle \Big|_{m=m'}^{l=l'} = \sum_l \frac{l(l+1)}{(2l-1)(2l+3)} & \left\{ (2l+1)[l(l+1) + (l^2 + l - 3)b^0 b^0] P_l(\mathbf{n} \cdot \mathbf{n}') \right. \\ & \left. - 8\pi(l^2 + l - 3) \sum_m (b^0 b^0 + b^+ b^-) |\mathbf{Y}_{lm}^l(\mathbf{n})|^{0*} |\mathbf{Y}_{lm}^l(\mathbf{n}')|^0 \right\} I_d^{(l,l)}. \quad (\text{B3}) \end{aligned}$$

The second type of term is the $l = l'$ and $m = m' \pm 1$ correlation proportional to $\sin \Theta_B \cos \Theta_B e^{\pm i\phi_B}$ on the right hand side of Eq. (30), which results in

$$\begin{aligned} \left\langle \frac{\Delta T}{T}(\mathbf{n}) \frac{\Delta T}{T}(\mathbf{n}') \right\rangle \Big|_{m'=m \pm 1}^{l=l'} = \sum_{l,m} \frac{2l(l+1)(l^2 + l - 3)}{(2l-1)(2l+3)} & \left\{ b^0 b^+ [|\mathbf{Y}_{lm}^l(\mathbf{n})|^{0*} |\mathbf{Y}_{lm}^l(\mathbf{n}')|^- + |\mathbf{Y}_{lm}^l(\mathbf{n})|^- |\mathbf{Y}_{lm}^l(\mathbf{n}')|^{0*}] \right. \\ & \left. + b^0 b^- [|\mathbf{Y}_{lm}^l(\mathbf{n})|^{0*} |\mathbf{Y}_{lm}^l(\mathbf{n}')|^+ + |\mathbf{Y}_{lm}^l(\mathbf{n})|^+ |\mathbf{Y}_{lm}^l(\mathbf{n}')|^{0*}] \right\} I_d^{(l,l)}, \quad (\text{B4}) \end{aligned}$$

where we have used Eqs. (A13). The third type of term is the $l = l'$ and $m = m' \pm 2$ correlation proportional to $\sin^2 \Theta_B e^{\pm 2i\phi_B}$ on the right side of Eq. (30), which results in⁷

$$\begin{aligned} \left\langle \frac{\Delta T}{T}(\mathbf{n}) \frac{\Delta T}{T}(\mathbf{n}') \right\rangle \Big|_{m'=m \pm 2}^{l=l'} = & \\ \sum_{l,m} \frac{2l(l+1)(l^2 + l - 3)}{(2l-1)(2l+3)} & \left\{ b^+ b^+ |\mathbf{Y}_{lm}^l(\mathbf{n})|^{+*} |\mathbf{Y}_{lm}^l(\mathbf{n}')|^- + b^- b^- |\mathbf{Y}_{lm}^{l-1}(\mathbf{n})|^{-*} |\mathbf{Y}_{lm}^{l+1}(\mathbf{n}')|^+ \right\} I_d^{(l,l)}. \quad (\text{B5}) \end{aligned}$$

$I_d^{(l,l)}$ in all three of these equations is defined in Eq. (29). Combining the expressions in Eqs. (B3)–(B5), we obtain

$$\begin{aligned} 4\pi \left\langle \frac{\Delta T}{T}(\mathbf{n}) \frac{\Delta T}{T}(\mathbf{n}') \right\rangle \Big|_{l=l'}^{l=l'} = \sum_l \frac{l(l+1)}{(2l-1)(2l+3)} & \left\{ (2l+1)(2l^2 + 2l - 3) P_l(\mathbf{n} \cdot \mathbf{n}') \right. \\ & \left. - 4\pi(l^2 + l - 3) \sum_m \left[(\mathbf{b} \cdot \mathbf{Y}_{lm}^l(\mathbf{n}))^* (\mathbf{b} \cdot \mathbf{Y}_{lm}^l(\mathbf{n}')) + (\mathbf{b} \cdot \mathbf{Y}_{lm}^l(\mathbf{n})) (\mathbf{b} \cdot \mathbf{Y}_{lm}^l(\mathbf{n}'))^* \right] \right\} I_d^{(l,l)}. \quad (\text{B6}) \end{aligned}$$

There are three types of off-diagonal terms in Eq. (B2) (See Eq. (38)), similar to the diagonal case classified just above. The first type of term is the $l = l' \pm 2$ and $m = m'$ correlation proportional to $3 \cos^2 \Theta_B - 1$ on the right hand

⁷ For symmetry reasons we have used correlations evaluated for $m \mp 1 = m' \pm 1$, and not for $m = m' \pm 2$. Thus in the expressions in Eqs. (34) we replace m by $m \pm 1$. Of course, this does not affect the final results; it just makes the computations easier and more symmetric.

side of Eq. (38), which results in

$$\left\langle \frac{\Delta T}{T}(\mathbf{n}) \frac{\Delta T}{T}(\mathbf{n}') \right\rangle \Big|_{m=m'}^{l=l' \pm 2} = \sum_{l,m} \frac{(l+2)(l-1)\sqrt{l(l+1)}}{2(2l+1)} (b^0 b^0 + b^+ b^-) \left\{ |\mathbf{Y}_{lm}^{l-1}(\mathbf{n})|^{0*} |\mathbf{Y}_{lm}^{l+1}(\mathbf{n}')|^0 \right. \\ \left. + |\mathbf{Y}_{lm}^{l+1}(\mathbf{n})|^{0*} |\mathbf{Y}_{lm}^{l-1}(\mathbf{n}')|^0 + |\mathbf{Y}_{lm}^{l-1}(\mathbf{n})|^0 |\mathbf{Y}_{lm}^{l+1}(\mathbf{n}')|^{0*} + |\mathbf{Y}_{lm}^{l+1}(\mathbf{n})|^0 |\mathbf{Y}_{lm}^{l-1}(\mathbf{n}')|^{0*} \right\} I_d^{(l-1, l+1)}. \quad (\text{B7})$$

Here we have used the relations $\mathbf{b} \cdot \mathbf{b}^* = b^0 b^{*0} + b^+ b^{*+} + b^- b^{*-} = (b^0)^2 - 2b^+ b^- = 1$, $(\mathbf{Y}_{lm}^L)^* = (-1)^{L+l+m+1} \mathbf{Y}_{l,-m}^L$, and $|(\mathbf{Y}_{lm}^L)^*|^\mu = |(\mathbf{Y}_{lm}^L)|_\mu$. The second type of term is the $l = l' \pm 2$ and $m = m' \pm 1$ correlation proportional to $\sin^2 \Theta_B \cos \Theta_B e^{\pm i\phi_B}$ on the right hand side of Eq. (38), which results in

$$\left\langle \frac{\Delta T}{T}(\mathbf{n}) \frac{\Delta T}{T}(\mathbf{n}') \right\rangle \Big|_{m'=m \pm 1}^{l=l' \pm 2} = - \sum_{l,m} \frac{(l+2)(l-1)\sqrt{l(l+1)}}{(2l+1)} I_d^{(l-1, l+1)} \\ \times \left\{ b^0 b^+ \left[|\mathbf{Y}_{lm}^{l-1}(\mathbf{n})|^{0*} |\mathbf{Y}_{lm}^{l+1}(\mathbf{n}')|^- + |\mathbf{Y}_{lm}^{l+1}(\mathbf{n})|^{0*} |\mathbf{Y}_{lm}^{l-1}(\mathbf{n}')|^- - |\mathbf{Y}_{lm}^{l-1}(\mathbf{n})|^0 |\mathbf{Y}_{lm}^{l+1}(\mathbf{n}')|^{+*} - |\mathbf{Y}_{lm}^{l+1}(\mathbf{n})|^0 |\mathbf{Y}_{lm}^{l-1}(\mathbf{n}')|^{+*} \right] \right. \\ \left. + b^0 b^- \left[|\mathbf{Y}_{lm}^{l-1}(\mathbf{n})|^{0*} |\mathbf{Y}_{lm}^{l+1}(\mathbf{n}')|^+ + |\mathbf{Y}_{lm}^{l+1}(\mathbf{n})|^{0*} |\mathbf{Y}_{lm}^{l-1}(\mathbf{n}')|^+ + |\mathbf{Y}_{lm}^{l-1}(\mathbf{n})|^0 |\mathbf{Y}_{lm}^{l+1}(\mathbf{n}')|^{-*} - |\mathbf{Y}_{lm}^{l+1}(\mathbf{n})|^0 |\mathbf{Y}_{lm}^{l-1}(\mathbf{n}')|^{-*} \right] \right\}. \quad (\text{B8})$$

Here we used Eqs. (A13). The third type of term is the $l = l' \pm 2$ and $m = m' \pm 2$ correlation proportional to $\sin^2 \Theta_B e^{\pm 2i\phi_B}$, on the right hand side of Eq. (38), which results in (see footnote 6)

$$\left\langle \frac{\Delta T}{T}(\mathbf{n}) \frac{\Delta T}{T}(\mathbf{n}') \right\rangle \Big|_{m'=m \pm 2}^{l=l' \pm 2} = - \sum_{l,m} \frac{(l+2)(l-1)\sqrt{l(l+1)}}{2(2l+1)} I_d^{(l-1, l+1)} \\ \times \left\{ b^+ b^+ \left[|\mathbf{Y}_{lm}^{l-1}(\mathbf{n})|^{+*} |\mathbf{Y}_{lm}^{l+1}(\mathbf{n}')|^- + |\mathbf{Y}_{lm}^{l+1}(\mathbf{n})|^{+*} |\mathbf{Y}_{lm}^{l-1}(\mathbf{n}')|^- + |\mathbf{Y}_{lm}^{l-1}(\mathbf{n})|^+ |\mathbf{Y}_{lm}^{l+1}(\mathbf{n}')|^{-*} + |\mathbf{Y}_{lm}^{l+1}(\mathbf{n})|^+ |\mathbf{Y}_{lm}^{l-1}(\mathbf{n}')|^{-*} \right] \right. \\ \left. + b^- b^- \left[|\mathbf{Y}_{lm}^{l-1}(\mathbf{n})|^{-*} |\mathbf{Y}_{lm}^{l+1}(\mathbf{n}')|^+ + |\mathbf{Y}_{lm}^{l+1}(\mathbf{n})|^{-*} |\mathbf{Y}_{lm}^{l-1}(\mathbf{n}')|^+ + |\mathbf{Y}_{lm}^{l-1}(\mathbf{n})|^- |\mathbf{Y}_{lm}^{l+1}(\mathbf{n}')|^{+*} + |\mathbf{Y}_{lm}^{l+1}(\mathbf{n})|^- |\mathbf{Y}_{lm}^{l-1}(\mathbf{n}')|^{+*} \right] \right\}. \quad (\text{B9})$$

In these expressions $I_d^{(l,l')}$ is given in Eq. (24). Combining the expressions in Eqs. (B7)–(B9) and taking into account that $\langle \Delta T/T(\mathbf{n}) \Delta T/T(\mathbf{n}') \rangle = \langle \Delta T/T(\mathbf{n}) \Delta T/T(\mathbf{n}') \rangle^*$ we obtain

$$\left\langle \frac{\Delta T}{T}(\mathbf{n}) \frac{\Delta T}{T}(\mathbf{n}') \right\rangle \Big|_{l=m}^{l=l' \pm 2} = \sum_{l,m} \frac{(l+2)(l-1)\sqrt{l(l+1)}}{2(2l+1)} I_d^{(l-1, l+1)} \left\{ (\mathbf{b} \cdot \mathbf{Y}_{lm}^{l-1}(\mathbf{n}))^* (\mathbf{b} \cdot \mathbf{Y}_{lm}^{l+1}(\mathbf{n}')) \right. \\ \left. + (\mathbf{b} \cdot \mathbf{Y}_{lm}^{l+1}(\mathbf{n}))^* (\mathbf{b} \cdot \mathbf{Y}_{lm}^{l-1}(\mathbf{n}')) + (\mathbf{b} \cdot \mathbf{Y}_{lm}^{l-1}(\mathbf{n})) (\mathbf{b} \cdot \mathbf{Y}_{lm}^{l+1}(\mathbf{n}'))^* + (\mathbf{b} \cdot \mathbf{Y}_{lm}^{l+1}(\mathbf{n})) (\mathbf{b} \cdot \mathbf{Y}_{lm}^{l-1}(\mathbf{n}'))^* \right\}. \quad (\text{B10})$$

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